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The geometry of null rotation identifications

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Abstract: The geometry of flat spacetime modded out by a null rotation (boost + rotation) is analysed. When embedding this quotient spacetime in String/M-theory, it still preserves one half of the original supersymmetries. Its connection with the BTZ black hole, supersymmetric dilatonic waves and one possible resolution of its singularity in terms of nullbranes are also discussed.

Keywords: Superstring Vacua, Classical Theories of Gravity, Supersymmetry and Duality.

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1. Introduction

If string theory provides us with a quantum theory of gravity, it is natural to study the propagation of strings in time dependent backgrounds. Some of the easiest models one can think of are constructed as lorentzian orbifolds of Minkowski spacetime. These have recently received considerable attention [1]–[5]. See also [6] for related work. The purpose of this work is to analyse the geometry of Minkowski spacetime modded out by a null rotation, what one may call null orbifold. One can look at this quotient space as some intermediate case between modding out by a rotation and modding out by a boost, which is still singular, but preserves one half of the spacetime supersymmetries. This set up was already considered in [7], and also in the context of twisting the $N = 2$ string [8].

The first, and original, motivation for the present work was provided by the cosmological scenario presented in [2]. They found a classical solution to Einstein equations coupled to a single scalar field subject to no potential with two branches. The first branch ($t < 0$) describes a contracting universe, whereas the second one ($t > 0$), an expanding universe. They are connected through a singularity at $t = 0$, where the Planck scale vanishes and the general relativity description breaks down. When embedded in string theory, the limit $t \to 0$ corresponds to a weak coupling regime, and the vanishing of the Planck scale tells us that higher dimensional operators become important as $t \to 0$. Thus, a perturbative worldsheet description is appropiate and reliable near the singularity. A fascinating remark was also given in [2] in order to deal with such a worldsheet description. It was stated there that the d-dimensional configuration could be understood as simply $\mathbb{R}^{d-1} \times \mathcal{M}^2$ where $\mathcal{M}^2$ is the two dimensional Milne universe. The fact that $\mathcal{M}^2$ can be understood as the quotient of the interior of the past and future light cones by the action of the group $\mathbb{Z}$ generated

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by a boost suggests that the two dimensional worldsheet field theory might be that of a lorentzian orbifold of Minkowski spacetime. This possibility has been recently studied in [4, 5]. In [4], a possible resolution of the lorentzian orbifold by adding a transverse compact spacelike circle was also considered.

Motivated by the above construction, we shall look for a spacetime geometry that can be thought of Minkowski spacetime modded out by the action of some vector field acting non-trivially on time, such that the quotient manifold remains supersymmetric (modding out by a boost breaks supersymmetry completely) and has no closed timelike curves (CTCs). It turns out such a spacetime geometry is unique:[1] it corresponds to Minkowski spacetime in three dimensions modded out by a null rotation. A null rotation in the x^1 direction is infinitesimally generated by the Killing vector

$$\xi_{\rm null} = B_{01} \pm R_{12} = x^0\partial_1 + x^1\partial_0 \pm \left(x^1\partial_2 - x^2\partial_1\right) ,$$

so that it consists of a boost in the x^1 direction plus a rotation in the 12-plane such that both the rapidity of the boost and the angle of the rotation have the same norm. Thus, as stressed before, this is a particular modding in the family parametrised by $\xi = \alpha B_{01} + \beta R_{12}$. Whenever $|\beta| > |\alpha|$, there always exists a Lorentz transformation such that $\xi = R_{12}$, whereas when $|\alpha| > |\beta|$, this freedom under conjugations allows us to write $\xi = B_{01}$. The norm of the Killing vector $\xi_{\rm null}$ vanishes at $x^- = 0$, where our quotient manifold is even not Hausdorff, due to the existence of fixed points ($x^- = x^1 = 0$). For $x^- \neq 0$, the spacetime looks like a strip of length proportional to $|x^-|$ with boundaries satisfying non-trivial identification conditions. This geometry and the absence of CTCs are discussed in section 2. Notice that the existence of a covariantly constant null vector in this spacetime does not allow us to view the singularity at $x^- = 0$ as formed to the future of a non-singular surface. In that respect, it looks closer to a singular wave solution [1].

The fact that our modding involves both boosts and rotations raises the question whether the above spacetime geometry is related, in some way, with the BTZ black hole [9, 10]. This is discussed in section 3, where it is shown that under a double scaling limit, the BTZ geometry reduces to the null rotation geometry.

The second motivation for this work was to understand the geometry behind the null-brane solution found in [11]. Nullbranes were obtained by Kaluza-Klein reduction along the orbits of the Killing vector

$$\xi = R\,\partial_z + \xi_{\rm null} ,$$

where z stands for a compact transverse spacelike coordinate of length R. Since $|\xi| > 0$ everywhere and it has no fixed points, such a spacetime modding constitutes one possible way of resolving the singularity found in the null rotation geometry. It is indeed proved that such spacetime has no closed causal curves. This is discussed in section 4, where we also include the singular ten dimensional configuration, the so called dilatonic wave, whose uplift to eleven dimensions can be interpreted as the null rotation geometry. Some comments concerning the duality relations among nullbranes and dilatonic waves are also

[1] The possibility of a compact timelike direction ($\xi = \partial_t$) or adding further transverse rotations to $\xi_{\rm null}$ are not considered in this paper.

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considered. We conclude with some extensions of the construction presented in this paper to curved backgrounds, trying to emphasize the universality of this new sector in string theory.

Note added. When this work was being completed, we learnt of some work in progress [12] in which the same model is analysed.

2. Null rotation identifications

Given a minkowskian spacetime in d+1 dimensions, any one parameter subgroup generated by a Killing vector ξ acting on this space

$$\mathrm{P} \to e^{\xi}\,\mathrm{P}$$

defines a quotient space by identifying points along its orbit. In the present case, ξ can be decomposed as

$$\xi = \tau + \lambda\,,$$

where $\tau \in \mathfrak{t} = \mathbb{R}^{1,d}$ is a translation and $\lambda \in \mathfrak{l} = \mathfrak{so}(1,d)$ is a Lorentz transformation. By conjugating with a Lorentz transformation one can bring λ to a normal form. Since we are just interested in one parameter subgroups acting non-trivially on time, one is left with two possiblities

$$\lambda = B_{01}(\gamma) + R_{\perp}(\beta)$$
$$\lambda = N_{+1}(u) + R_{\perp}(\beta)\,,$$

where $B_{01}(\gamma)$ is an infinitesimal boost with parameter γ along direction 1, $N_{+1}(u) \equiv B_{01}(u) \pm R_{12}(u)$ is a null rotation with parameter u in the direction 1 and R stands for rotations. In the following, we shall not consider the possibility of rotating in the transverse directions ($\beta = 0$).

Preservation of supersymmetry excludes boosts and selects null rotations. This can be easily proved by studying how Killing vectors ξ act on Killing spinors ε through the spinorial Lie derivative [13]

$$\mathcal{L}_{\xi}\varepsilon = \nabla_{\xi}\varepsilon + \frac{1}{4}\partial_{[a}\xi_{b]}\Gamma^{ab}\varepsilon\,.$$

There is a second possibility for acting non-trivially on time while preserving full supersymmetry, which is by considering a time translation, $\xi = \tau = \partial_t$, but since we are not willing to consider compact time directions, we shall concentrate on null rotation identifications, which preserve one half of the supersymmetries.

2.1 Geometry

One way to describe the geometry of the null rotation identifications is by fibering the original spacetime with the orbits of the associated Killing vector and studying the effects of the identifications in this adapted coordinate system. The orbits are given by

$$\begin{aligned} x^{-}(s) &= x_0^{-} \\ x^{+}(s) &= x_0^{+} + 2x_0^{1}\,s + x_0^{-}\,s^2 \\ x^{1}(s) &= x_0^{1} + x_0^{-}\,s \end{aligned} \tag{2.1}$$

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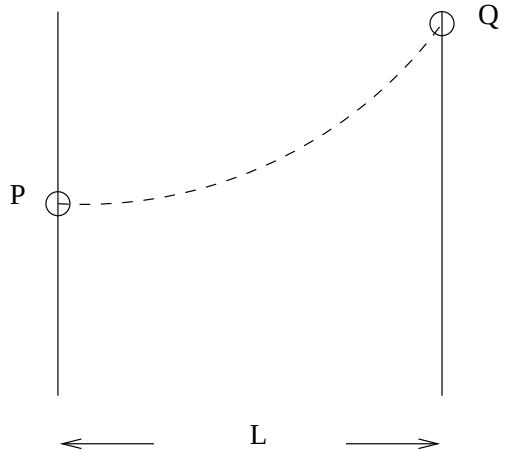

Figure 1: Geometry of the $x^- > 0$ sections of flat spacetime after null rotation identifications. The length of the strip is $L = x^- \cdot \beta$. Points P and Q are identified. The dashed line stands for the original orbit of the Killing vector ξ_{null}.

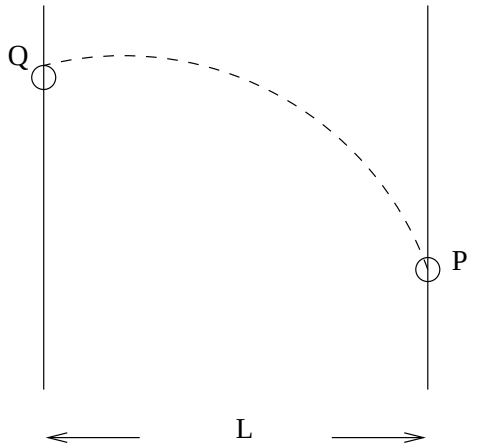

Figure 2: Geometry of the $x^- < 0$ section of flat spacetime after null rotation identifications. The length of the strip is $L = |x^-| \cdot \beta$. Points P and Q are identified. The dashed line stands for the original orbit of the Killing vector ξ_{null}.

where x_0^i are initial conditions, s is the affine parameter of the orbit and $x^\pm = x^0 \pm x^2$ are lightlike coordinates. Notice that the line located at $x^- = x^1 = 0$ is a line of fixed points.

There are two reasons why to study sections of spacetime with constant x^-. First of all, ξ_{null} does not act on x^-. Furthermore, the norm of the Killing vector, $|\xi_{\text{null}}|^2 = (x^-)^2$, teaches us to consider either $x^- \neq 0$ or $x^- = 0$. This is precisely what we shall do next.

Let us start on the region where $x^- \neq 0$. The plane spanned by $\{x^+ , x^1\}$ is fibered by $\{s , x_0^+\}$, since as we vary x_0^+ and move along the fiber (s) one covers the full subspace. We are interested in the geometry of spacetime when non-trivial identifications are set along the orbits (2.1), that is

$$s \sim s + \beta \,,$$

for some dimensionless parameter β. The first consequence of such an identification is that spacetime becomes a strip of length $L = x^- \cdot \beta$ along the x^1 direction. We shall use the freedom to locate the strip to set x_0^1 to zero. The x^+ direction remains a non-compact one, but due to the parabolic nature of $x^+(s)$, the boundaries of the forementioned strip satisfy non-trivial identification conditions. Indeed, the point $P \equiv (x_0^+ , 0)$ is identified with $Q \equiv (x_0^+ + L\,\beta , L)$

$$P \equiv (x_0^+ , 0) \sim (x_0^+ + L\,\beta , L) \equiv Q \quad \forall\, x_0^+ \,.$$

This geometry is illustrated in figures 1 and 2 for $x^- > 0$ and $x^- < 0$, respectively.

As $|x^-|$ decreases, the paraboles start degenerating and the strip shrinks. In fact, the limit $x^- \to 0$ is singular, as can be seen from different points of view. First of all, the space at $x^- = 0$ is not Haussdorf, due to the existence of fixed points. Its fibering is no longer given in terms of $\{s , x_0^+\}$, but in terms of $\{s , x_0^1\}$. This can be easily derived from the orbits (2.1). Indeed, x^1 is left invariant at $x^- = 0$, thus it remains non-compact in that

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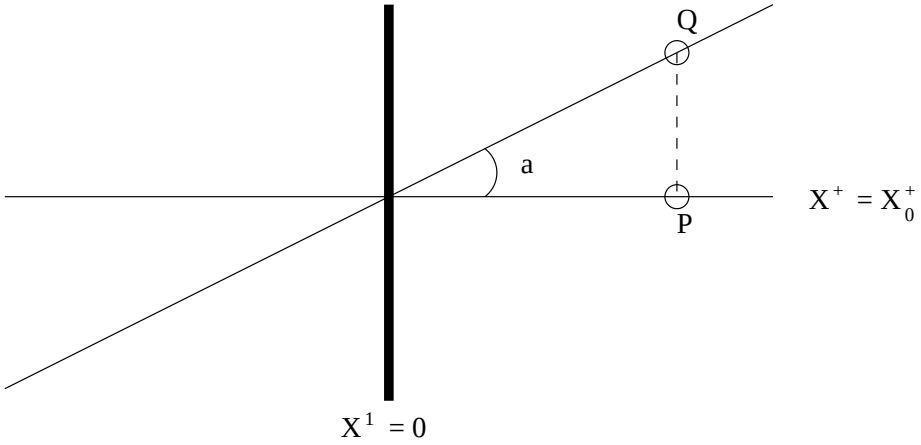

Figure 3: Geometry of the $x^- = 0$ section of flat spacetime after null rotation identifications. The bold line at $x^1 = 0$ stands for the line of fixed points. Points P and Q are identified. The angle a is determined by $\tan a = 2\beta$. The dashed line stands for a closed null curve.

subspace. On the other hand, the previous paraboles degenerate into a line of fixed points located at $x^1 = 0$. Away from $x^1 = 0$, one is left with $\mathbb{R}^2$ where points $P \equiv (x_0^+ , x_0^1)$ are identified with points $Q \equiv (x_0^+ + 2x_0^1 \cdot \beta , x_0^1)$

$$P \equiv (x_0^+ , x_0^1) \sim (x_0^+ + 2x_0^1 \cdot \beta , x_0^1) \equiv Q \qquad \forall\, x_0^1 \neq 0 \,.$$

The geometry of the $x^- = 0$ subspace is illustrated in figure 3. One is thus left with two regions of spacetime ($x^- > 0$ and $x^- < 0$) connected through $x^- = 0$, where the quotient space is not even a Haussdorf space. In the next subsection, we shall analytically prove the existence of closed null curves. These can easily be guessed by looking at figure 3 and considering the null curves (dashed line on the figure),

$$x^- = 0 \,, \qquad x^1 = \text{constant}$$

connecting points P and Q which are identified by the action of the U(1) subgroup generated by ξ_{null}. Thus, at $x^- = 0$, spacetime has also causal singularities.

At this stage, one can just remove $x^- = 0$ from spacetime, leaving two disconnected regions with no causal singularities. A more sensible possibility is to embed such scenario in string theory and analyse whether the twisted sectors located at $x^- = 0$ manage to resolve such singularity. A further possibility to smooth it is to modify the action of the U(1) subgroup by adding a transverse compact spacelike direction

$$\xi = \xi_{\text{null}} \to R\partial_z + \xi_{\text{null}} \,.$$

We postpone the discussion of such a resolution until we embed this geometry in string theory in section 4.

2.2 Closed causal curves

A necessary condition for the absence of closed causal curves in quotient spaces of the type discussed above is that the norm of the Killing vector generating the U(1) subgroup action

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is strictly positive. Such a property is not satisfied by $\xi_{\rm null}$, thus closed null curves are expected at $x^- = 0$, as already stressed before. The question we would like to address is the non-existence of closed causal curves for $x^- \neq 0$. This is non-trivial, because the forementioned criterium is not a sufficient condition.

The proof uses an adapted coordinate system $\{t\,,x\,,y\}$ to the action of $\xi_{\rm null}$ ($\xi_{\rm null} = \partial_y$) which is valid for $x^- \neq 0$, the subspace we are interested in. This coordinate system is defined by

$$\begin{aligned} x^- &= t + x \\ x^+ &= t - x + (t+x)y^2 \\ x^1 &= (t+x)y\,, \end{aligned} \tag{2.2}$$

in which the three dimensional metric can be written as

$$g = -(dt)^2 + (dx)^2 + (t+x)^2(dy)^2\,. \tag{2.3}$$

We want to show there is no causal curve $x^M(\lambda)$, i.e. $\frac{dx^M}{d\lambda}\frac{dx^N}{d\lambda}g_{MN}(x) \leq 0\ \forall\,\lambda$, connecting the points $(t_0\,,x_0\,,y_0)$ and $(t_0\,,x_0\,,y_0+\beta)$. If we assume that such a curve exists, there must necessarily exist one value of the affine parameter $\lambda = \lambda^\star$ where $dt/d\lambda$ vanishes

$$\exists\lambda = \lambda^\star \qquad \text{s.t.} \qquad \left(\frac{dt}{d\lambda}\right)_{\lambda^\star} = 0\,.$$

The norm of the tangent vector to $x^M(\lambda)$ at $\lambda = \lambda^\star$ is

$$\left(\frac{dx}{d\lambda}\right)^2_{\lambda^\star} + (t+x)^2(\lambda^\star)\left(\frac{dy}{d\lambda}\right)^2_{\lambda^\star} \leq 0\,.$$

Such an inequality can never be satisfied for a timelike curve, so no closed timelike curves exist. On the other hand, the norm vanishes when

$$\left(\frac{dx}{d\lambda}\right)_{\lambda^\star} = \left(\frac{dy}{d\lambda}\right)_{\lambda^\star} = 0\,,$$

since we are away from $t + x = 0$. Thus, at $\lambda = \lambda^\star$, the tangent vector to the causal curve vanishes identically, which contradicts λ being an affine parameter. We conclude that closed causal curves are not allowed when $t + x = x^- \neq 0$.

3. Relation with the BTZ black hole

The BTZ black hole line element [9] is given by

$$g_{(3)} = -f^2(r)(dt)^2 + f^{-2}(r)(dr)^2 + r^2\left[d\phi - \frac{J}{2r^2}dt\right]^2\,, \tag{3.1}$$

the lapse function being defined by

$$f^2(r) = -M + \left(\frac{r}{l}\right)^2 + \frac{J^2}{4r^2}\,,$$

where M and J are two constants of integration which are identified as the mass and angular momentum.

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This is a solution to Einstein field equations in three dimensions with negative cosmological constant Λ related to the radius of curvature through $-\Lambda = l^{-2}$. The lapse function vanishes for two values of r given by

$$r_{\pm} = l \left[\frac{M}{2} \left\{ 1 \pm \left[1 - \left(\frac{J}{M \cdot l} \right)^2 \right]^{1/2} \right\} \right]^{1/2} , \tag{3.2}$$

whereas g_{tt} vanishes at

$$r_{\rm erg} = l \cdot M^{1/2} .$$

These three special values of r obey

$$r_- \leq r_+ \leq r_{\rm erg} .$$

As it happens in $3+1$ dimensions for the Kerr metric, r_+ is the black hole horizon, $r_{\rm erg}$ is the surface of infinite redshift, and the region between r_+ and $r_{\rm erg}$ is the ergosphere. The solution describes a black hole whenever

$$M > 0 , \qquad |J| \leq M \cdot l .$$

Besides the continuous black hole spectrum above the vacuum ($M = J = 0$), there is a sort of "bound state" space separated from the vacuum by a mass gap of one unit ($M = -1$ and $J = 0$), which can not be continuously deformed to the vacuum and has neither singularities nor horizon. This is anti-de Sitter space, which can be defined in terms of its embedding in a four-dimensional flat space of signature $(- - ++)$

$$g_{(4)} = -(du)^2 - (dv)^2 + (dx)^2 + (dy)^2$$

through the equation

$$-v^2 - u^2 + x^2 + y^2 = -l^2 .$$

A system of coordinates covering the whole manifold may be introduced by setting

$$\begin{aligned} u &= l \cosh\mu \sin\lambda , \qquad & v &= l \cosh\mu \cos\lambda \qquad & 0 \leq \mu < \infty \\ x &= l \sinh\mu \cos\theta , \qquad & y &= l \sinh\mu \sin\theta \qquad & 0 \leq \lambda, 2\pi \end{aligned} \tag{3.3}$$

which yields

$$g_{\rm adS} = l^2 \left[-\cosh^2\mu (d\lambda)^2 + (d\mu)^2 + \sinh^2\mu (d\theta)^2 \right] . \tag{3.4}$$

Notice that strictly speaking, by an abuse of language, one refers to anti-de Sitter space to its universal covering space, the one in which the angular coordinate λ has been "unwrapped", to avoid the existence of closed timelike curves (CTCs).

By construction, the anti-de Sitter metric is invariant under $\mathrm{SO}(2,2)$. The Killing vectors generating $\mathfrak{so}(2,2)$ are explicitly given below for later reference

$$\begin{aligned} J_{01} &= v\partial_u - u\partial_v , \qquad & J_{02} &= x\partial_v + v\partial_x , \\ J_{03} &= y\partial_v + v\partial_y , \qquad & J_{12} &= x\partial_u + u\partial_x , \\ J_{13} &= y\partial_u + u\partial_y , \qquad & J_{23} &= y\partial_x - x\partial_y . \end{aligned}$$

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It was shown in [10] that (3.1) can be obtained by making identifications along the orbits generated by the Killing vector

$$\xi_{\rm BTZ} = \frac{r_+}{l} J_{12} - \frac{r_-}{l} J_{03} - J_{13} + J_{23} \,. \tag{3.6}$$

The latter contains boosts and rotations, so it is rather natural to ask whether there is some relation between the null rotation identification space and the BTZ black hole. It will be proved that the former appears as a double scaling limit of the latter.[2]

Just by looking at the construction of both spaces, the strategy of the proof emerges naturally. Both spaces are constructed by modding out an starting spacetime by the action of a certain one dimensional subgroup generated by a given Killing vector. Thus, if one establishes a precise map between the starting configurations and the Killing vectors used in the modding, the relation among both quotient spaces will have been established.

The starting configurations are maximally symmetric spaces of vanishing and negative cosmological constant, respectively. It is thus a necessary condition, but not sufficient, to study the limit $l \to \infty$, in which one is just left with the inside of the black hole (the exterior is pushed away to infinity). If one further concentrates on a region pretty close to the origin of spacetime by considering the double scaling limit

$$\lambda \to \frac{t}{l} \,, \qquad \mu \to \frac{r}{l} \,, \qquad l \to \infty \tag{3.7}$$

it is easy to check that the anti-de Sitter metric is mapped into the Minkowski metric and the Lie algebra $\mathfrak{so}(2,2)$ contracts to $\mathfrak{so}(1,2) \ltimes \mathbb{R}^{1,2}$.

Notice that the radial coordinate $r = l \sinh \mu$ in the BTZ black hole metric (3.1) does not scale in the limit. This means that if $l \sim \epsilon^{-a}$ $(a > 0)$, in the limit $\epsilon \to 0$

$$r_\pm \quad \text{fixed} \,, \qquad M \sim \epsilon^{2a} \,, \qquad J \sim \epsilon^a \,. \tag{3.8}$$

Thus, the double scaling limit relating both maximally symmetric spaces enforces the mass and angular momentum to vanish, keeping $r_\pm$ fixed. Since the black hole interpretation disappears, r_+ no longer corresponds to a horizon.

When one analyses the Killing vector (3.6) in the double scaling limit (3.7) and (3.8), it reduces to

$$\xi_{\rm BTZ} \to -r_- \partial_y - B_{0y} + R_{yx} \,,$$

which is equivalent to the one used in the null rotation construction. Indeed, just by a orientation reversal discrete transformation ($x \to -x$, $y \to -y$) and by a change of origin (conjugation under translations)

$$\xi_{\rm BTZ} \to N_{+y} = \xi_{\rm null} \,. \tag{3.9}$$

4. Embedding in string/M-theory

It is straightforward to embed the previous three dimensional geometry in string theory and M-theory. One just needs to consider the maximally supersymmetric flat vacua of

[2] The author would like to thank Y. Antebi and T. Volansky for discussions on this point.

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these theories and construct the quotient space by identifying points along the orbit of $\xi_{\rm null}$. As discussed in section 2, such a quotient spacetime will be supersymmetric. In this section, we shall concentrate on this construction for M-theory and type-IIA string theory, even though it can just as well be applied to type IIB. We shall first look for some ten dimensional singular spacetime, whose uplift to M-theory can be interpreted as the corresponding scenario described in section 2 when embedded in M-theory. Afterwards, we shall comment on a possible resolution of the singularity by adding an extra compact spacelike dimension transverse to the action of the null rotation. It will be proved that the corresponding quotient spacetime has no closed causal curves, and that the corresponding ten dimensional geometry is the nullbrane supersymmetric configuration found in [11]. Some comments on duality relations among these spacetimes will be added. We conclude with some brief discussion concerning generalizations of the previous constructions to arbitrary curved backgrounds having an $SO(1,2)$ isometry subgroup.

4.1 Dilatonic waves

As reviewed in the introduction, one of the features of the cosmological scenario discussed in [2] was its higher dimensional interpretation in terms of flat space modded out by a boost. In the same spirit, one may look for a singular spacetime whose singularity might be interpreted as the one discussed in section 2. It is natural to change coordinates from the cartesian ones $\{x^{\pm}\,, x^1\}$ to the adapted ones $\{u\,, v\,, x\}$

$$\begin{aligned} x^- &= u \\ x^+ &= v + u\, x^2 \\ x^1 &= u\, x \end{aligned} \tag{4.1}$$

where x stands for the compact coordinate along the orbit. In this coordinate system, the metric for the three dimensional subspace of spacetime where $SO(1,2)$ acts, is written as [7]

$$g = -du \cdot dv + u^2 (dx)^2\,. \tag{4.2}$$

The Killing vector $\xi_{\rm null}$ becomes ∂_x in the above coordinate system and its norm vanishes at $u = 0$. If one analyses the Kaluza-Klein reduction of the eleven dimensional Minkowski spacetime along the orbits of $\xi_{\rm null}$, one expects a ten dimensional configuration becoming singular precisely where the causal singularity was located in M-theory. Proceeding in this way, one derives a $\frac{1}{2}$-supersymmetric type-IIA configuration:

$$\begin{aligned} g &= |t+x|\, \left\{ ds^2(\mathbb{E}^8) + (dx)^2 - (dt)^2 \right\} \\ \Phi - \Phi_0 &= \frac{3}{2} \log |t+x|\,, \end{aligned} \tag{4.3}$$

where we used cartesian coordinates $\{t\,, x\}$ instead of the lightlike ones $\{u\,, v\}$.

Notice that the conformal factor $|t+x|$ does not disappear in the Einstein frame, where the metric is given by

$$g_E = |t+x|^{1/4} \left\{ ds^2(\mathbb{E}^8) + (dx)^2 - (dt)^2 \right\}.$$

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On the other hand, the energy-momentum tensor has the same form as the one for an electromagnetic field $T_{mn} = ck_m k_n$. Indeed, the one form $k_{(1)}$ equals $k = d\Phi$, and since the latter defines a null one form, the second contribution to the energy-momentum tensor, $g^{mn}\partial_m\Phi\partial_n\Phi$, vanishes. Thus, T_{mn} vanishes except for the components on the t-x plane

$$T_{tt} = T_{tx} = T_{xx} = \frac{9}{4}\frac{1}{|t+x|^2}\,.$$

Thus, as expected, the ten dimensional geometry has an essential singularity on $t + x = 0$, the lightlike hypersurface where the norm of the Killing vector used in the reduction vanishes. Notice that even though the scalar curvature vanishes, both the Riemann and energy-momentum tensors diverge on the singularity.

4.2 Resolution of the singularity: nullbranes

One way of resolving the causal singularity that appears when modding out spacetime by a null rotation is to add a compact spacelike transverse direction in the action of the U(1) subgroup used to identify points. In other words, one adds a transverse translation ($\tau_\perp$) to the null rotation Killing vector ($\xi_{\rm null}$) and mods out spacetime by the action of

$$\xi = \tau_\perp + \xi_{\rm null}\,.$$

Such a Killing vector has an strictly positive norm, thus suggesting the absence of closed causal curves. Furthermore, the presence of $\tau_\perp$ removes all previous fixed points left invariant by the action of ξ. To sum up, one expects a non-singular $\frac{1}{2}$- supersymmetric spacetime, since translations do not modify the supersymmetry analysis presented before.

By analysing the orbits of ξ, it is straightforward to figure out the geometry of the quotient spacetime. In figure 4, it is explicitly shown how the previous causal singularity at $x^- = 0$ is resolved by the translation $\tau_\perp$.

In order to prove that there are no closed causal curves in the quotient spacetime, it is useful to work in adapted coordinates. These were found in [11]. Here, we just write the eleven dimensional metric in such a coordinate system:

$$g = \Lambda\,(dz + A)^2 + 2du\,dv - \Lambda^{-1}x^2(du)^2 + \Lambda^{-1}(dx)^2 + 2ux\Lambda^{-1}du\,dx + ds^2(\mathbb{E}^7)\,, \qquad (4.4)$$

where

$$\begin{aligned} \Lambda &= 1 + u^2 \\ A_1 &= \Lambda^{-1}\,(u\,dx - x\,du)\,, \end{aligned} \qquad (4.5)$$

and z stands for the compact transverse spacelike direction ($\xi = \partial_z$).

What we want to prove is that there is no causal curve $x^M(\lambda)$, i.e. $\frac{dx^M}{d\lambda}\frac{dx^N}{d\lambda}g_{MN}(x) \leq 0$ $\forall\,\lambda$, connecting the points $(x_0^i\,, z)$ and $(x_0^i\,, z + R)$, because if so, that would give rise to a closed curve in the quotient spacetime. The basic step of the proof is, once more, to realise that for such a curve to exist, there must be some value of the affine parameter λ where the component of the tangent vector $du/d\lambda$ vanishes. Mathematically,

$$\exists\,\lambda = \lambda^\star \qquad \text{s.t.} \qquad \left(\frac{du}{d\lambda}\right)_{\lambda^\star} = 0\,.$$

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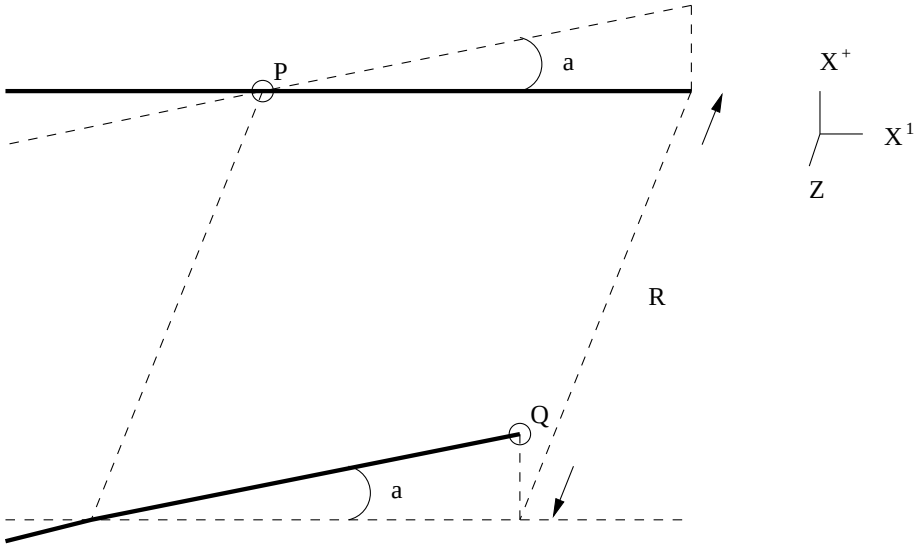

Figure 4: Resolution of the singularity at $x^- = 0$ by identifying points in spacetime through a compact translation plus a null rotation. Bold lines are identified, and in particular, points P and Q are identified.

If one evaluates the norm of the tangent vector to the *assumed* closed causal curve at $\lambda = \lambda^\star$

$$|\xi|^2 = \Lambda \left(\frac{dz}{d\lambda} + \Lambda^{-1} u \frac{dx}{d\lambda}\right)^2_{\lambda^\star} + \Lambda^{-1} \left(\frac{dx}{d\lambda}\right)^2_{\lambda^\star} + \sum_{i=1}^{7} \left(\frac{dx^i}{d\lambda}\right)^2_{\lambda^\star} ,$$

one appreciates that it is positive or zero, so one concludes there are certainly no closed timelike curves. Furthermore, the lower bound of the norm is saturated if and only if

$$\frac{dz}{d\lambda} = \frac{dx}{d\lambda} = \frac{dx^i}{d\lambda} = 0 \qquad \forall\, i \quad \text{at} \quad \lambda = \lambda^\star .$$

But such a possibility is excluded because λ is an affine parameter. Thus, closed null curves are also absent in this quotient spacetime.

Having resolved the singularity, we also conclude that the nullbrane configuration discovered in [11] is the ten dimensional configuration resolving the singularity of the dilatonic wave at $u = 0$.

We would like to finish this subsection with some remarks concerning the *local* relation among the different configurations discussed so far. As emphasized in [11, 14, 15, 16], any M-theory background with various identifications defines a moduli space of Kaluza-Klein reductions. Each point P in this moduli corresponds to a type-IIA configuration obtained by reduction along the orbits of a Killing vector field ξ_P. The freedom in choosing ξ_P is equivalent to the freedom in identifying the different circles in this background as the M-theory circle. Different type IIA configurations obtained in this way will all be U-dual to each other.[3] In the case at hand, our starting M-theory configuration is Minkowski spacetime with two different identifications (M-vacuum (a,b) in figure 5): along the S^1

[3] Actually, only the full backgrounds keeping all momentum modes are dual.

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circle, generated by $R\partial_z$, and along the null rotation, generated by $\xi_{\rm null}$. There are three different circles (Killing vector fields) that one can identify with the M-theory circle:

$$\begin{aligned}\xi_{(1)} &= R\partial_z \\ \xi_{(2)} &= \xi_{\rm null} \\ \xi_{(3)} &= R\partial_z + \xi_{\rm null}\,.\end{aligned}$$

Reduction along the orbits generated by $\xi_{(1)}$ [R(a) in figure 5] gives rise to Minkowski spacetime modded out by a null rotation [IIA-vacuum (b) in figure 5]. On the other hand, $\xi_{(2)}$ [R(b)] gives rise to a dilatonic wave with a transverse circle [Dilatonic wave (a)] and $\xi_{(3)}$ [R(a,b)] to the nullbrane configuration. All of them are connected to each other through the chain of dualities T S T', where T stands for T-duality and S for an $\mathrm{SL}(2\,,\mathbb{Z})$ transformation in type IIB, reminiscent of the $\mathrm{SL}(2\,,\mathbb{Z})$ symmetry of the torus spanned by $\{R\partial_z\,,\xi_{\rm null}\}$. This statement can be easily checked by starting with the nullbrane configuration and first changing coordinates from $\{u\,,v\,,x\}$ to $\{\tilde{u}\,,\tilde{v}\,,\tilde{x}\}$

$$\begin{aligned}\tilde{x} &= \frac{x}{u} - 1 \\ \tilde{u} &= u \\ \tilde{v} &= v - \frac{x^2}{u}\end{aligned} \tag{4.6}$$

in which the configuration is described by

$$\begin{aligned}g &= \Lambda^{1/2}\left\{-d\tilde{u}\cdot d\tilde{v} + ds^2(\mathbb{E}^7)\right\} + \Lambda^{-1/2}\tilde{u}^2(d\tilde{x})^2 \\ \Phi &= \Phi_0 + \frac{3}{4}\log\Lambda \\ A_{(1)} &= \Lambda^{-1}\tilde{u}^2 d\tilde{x}\,,\end{aligned} \tag{4.7}$$

where the scalar function Λ is defined by $\Lambda = 1 + \tilde{u}^2$. All the involved T-duality transformations are along the $\tilde{x}$ (or its dual directions). On the other hand, taking into account the most general $\mathrm{SL}(2\,,\mathbb{Z})$ transformation acting on $\lambda = C_{(0)} + \mathrm{i}e^{-\Phi}$, where $C_{(0)}$ is the RR zero form

$$\lambda' = \frac{a\lambda + b}{c\lambda + d} \qquad ad - bc = 1\,,$$

one can check that under the forementioned chain of dualities, the nullbrane configuration is mapped to:

(i) Minkowski spacetime with null rotation identifications for the choice $a = d = -c$, $b = 0$ and $d^2 = 1$.

(ii) Dilatonic wave in a transverse circle for the choice $a = c = -b$, $d = 0$ and $c^2 = 1$.

These duality maps are illustrated in figure 5.

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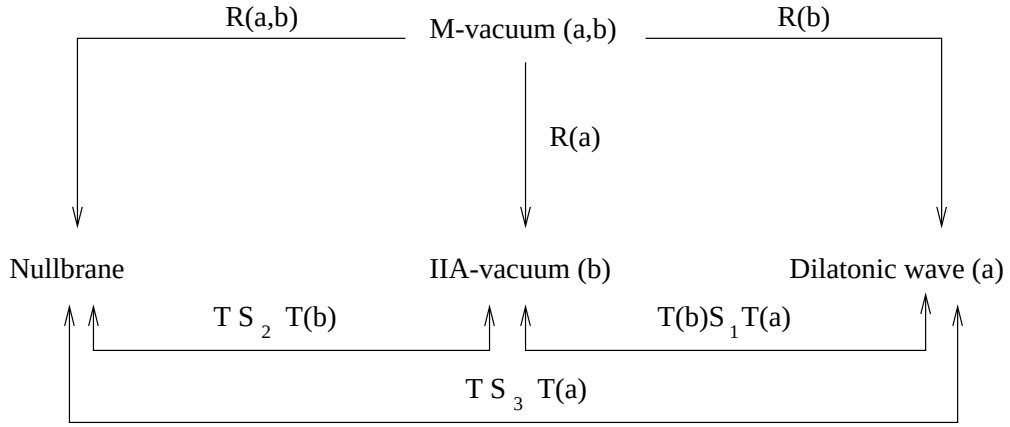

Figure 5: Duality relations among nullbranes, dilatonic waves and flat vacua with non-trivial identifications. R stands for Kaluza-Klein reduction, T for T-duality and S for an $SL(2\,,\mathbb{Z})$ transformation. a and b stand for the two different circles as explained in the text.

4.3 Extension to curved backgrounds

The purpose of the present subsection is to emphasize that the above construction is not relying on the flatness of the starting spacetime configuration, but on the existence of an $SO(1,2)$ isometry. Indeed, any spacetime compatible with string theory having an $SO(1,2)$ group as a subgroup of its isometry group allows such a construction. Thus, as stressed in [14], the existence of an $SO(1,2)$ symmetry and the possibility of identifying points along the null rotation subgroup, defines a new sector in string theory.

If one starts from the eleven dimensional M2-brane configuration, and reduces along the orbit of the Killing vector $\xi_{\rm null}$, one ends up with a dilatonic wave propagating on a fundamental string, which preserves one fourth of the spacetime supersymmetries, and is described by

$$\begin{aligned} g &= |t+x|\left\{U^{-1}[-(dt)^2+(dx)^2]+ds^2(\mathbb{E}^8)\right\} \\ \Phi-\Phi_0 &= \frac{3}{2}\log|t+x|-\frac{1}{2}\log U \\ H_{(3)} &= (t+x)dt\wedge dx\wedge dU^{-1}\,, \end{aligned} \tag{4.8}$$

where U is an harmonic function defined on $\mathbb{E}^8$.

Proceeding analogously with the eleven dimensional M5-brane configuration, one ends up with a dilatonic wave propagating on a D4-brane

$$\begin{aligned} g &= |t+x|\left\{U^{-1/2}[-(dt)^2+(dx)^2+ds^2(\mathbb{E}^2)]+ds^2(\mathbb{E}^5)\right\} \\ \Phi-\Phi_0 &= \frac{3}{2}\log|t+x|-\frac{1}{4}\log U \\ \star G_{(4)} &= (t+x)dt\wedge dx\wedge \mathrm{dvol}\,\mathbb{E}^2\wedge dU\,, \end{aligned} \tag{4.9}$$

where U is now defined on $\mathbb{E}^5$.

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If one considers the Mkk-monopole, one ends up with a dilatonic wave propagating on a KKA-monopole

$$\begin{aligned} g &= |t+x| \left\{ U^{-1/2}[-(dt)^2 + (dx)^2 + ds^2(\mathbb{E}^4)] + ds^2_{TN} \right\} \\ \Phi - \Phi_0 &= \frac{3}{2} \log|t+x| \end{aligned} \tag{4.10}$$

It is straightforward to extend these constructions to any background having an $SO(1,2)$ isometry.

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